

# REAL AND COMPLEX K-PLANES IN CONVEX HYPERSURFACES

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**ABSTRACT.** It is shown that the rank of the second fundamental form (resp. the Levi form) of a  $\mathcal{C}^2$ -smooth convex hypersurface  $M$  in  $\mathbb{R}^{n+1}$  (resp.  $\mathbb{C}^{n+1}$ ) does not exceed an integer constant  $k < n$  near a point  $p \in M$ , then through any point  $q \in M$  near  $p$  there exists a real (resp. complex)  $(n - k)$ -dimensional plane that locally lies on  $M$ .

It is a classical result in the differential geometry that any developable surface  $M$  in  $\mathbb{R}^3$  (i.e. with zero Gaussian curvature) is a part of a complete ruled surface (i.e. through every point of  $M$  there exists a straight line that lies on  $M$ ). Note that second fundamental form of such an  $M$  has rank 0 or 1 at any point. A similar result holds in higher dimensions (cf. [3, Lemma 2]):

(R) If the rank of the second fundamental form of a  $\mathcal{C}^2$ -smooth hypersurface  $M$  in  $\mathbb{R}^{n+1}$  is a constant  $k < n$  near a point  $p \in M$ , then  $M$  is locally generated by  $(n - k)$ -dimensional planes. (In particular, if  $k = 0$ , then  $M$  is locally a hyperplane.)

This result has a complex version (see [4, Theorem 6.1, Corollary 5.2]):

(C) If the rank of the Levi form of a  $\mathcal{C}^2$ -smooth real hypersurface  $M$  in  $\mathbb{C}^{n+1}$  is a constant  $k < n$  near a point  $p \in M$ , then  $M$  is locally foliated by complex  $(n - k)$ -dimensional manifolds. Moreover, if  $k = 0$  (i.e.  $M$  is Levi-flat) and  $M$  is real analytic, then  $M$  is locally biholomorphic to a complex hyperplane.

On the other hand, in both cases (real and complex), almost nothing is known if the rank is not maximal and non-constant.

The aim of this note is consider the last case when the hypersurface  $M$  is convex, i.e.  $M$  is a part of the boundary of convex domain.

**Proposition 1.** *The rank of the second fundamental form (resp. the Levi form) of a  $\mathcal{C}^2$ -smooth convex hypersurface  $M$  in  $\mathbb{R}^{n+1}$  (resp.  $\mathbb{C}^{n+1}$ ) does not exceed an integer constant  $k < n$  near a point  $p \in M$  if and*

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only if through any point  $q \in M$  near  $p$  there exists a real (resp. complex)  $(n - k)$ -dimensional plane that locally lies on  $M$ .

REMARK. If  $k = 0$  in the complex case, then  $M$  is locally linearly equivalent to the Cartesian product of  $\mathbb{C}^n$  and a planar domain (see [2, Theorem 1]).

*Proof.* If the respective real (complex)  $(n - k)$ -dimensional plane exists for a point  $q \in M$ , then the non-negativity of the second fundamental form (the Levi form) at  $q$  easily implies that the rank of the form at  $q$  does not exceed  $k$ .

For the converse, let first consider the complex case.

It is enough to show that through  $p$  there exists a complex line that locally lies on  $M$ . Then, considering the intersection of  $M$  with the orthogonal complement of this line, we may proceed by induction on  $n$  to find  $n - k$  orthogonal complex lines locally lying on  $M$ . The convexity of  $M$  easily implies that the  $(n - k)$ -dimensional planes, spanned by these lines, locally lies on  $M$ . Finally, note that the same holds for any point  $q \in M$  near  $p$  (since may replace  $p$  by  $q$ ).

Assume that there does not exist such a line. It is claimed in [6, p. 310] and proved in [5, Theorem 6] that  $p$  is a local holomorphic peak point for one of the sides, say  $M^+$ , of  $M$  near  $p$  (the convex one). By [1, Corollary 2],  $p$  is a limit of strictly pseudoconvex point of  $M^+$  which is a contradiction to the rank assumption.

The proof in the real case is similar. Recall that a point  $q \in M$  is called exposed if there exists a real hyperplane that intersects  $M$  in  $p$  alone (i.e.  $p$  is a linear peak point). It is enough to combine two facts:

- the set of exposed points is dense in the set of extreme points (see [7]);
- the set of strictly convex points of  $M^+$  (all the eigenvalues of the second fundamental form are positive) is dense in the set of exposed points.

The last fact can be shown following, for example, the proof of [1, Theorem].  $\square$

REMARK. The 'only if' part ( $\rightarrow$ ) of Proposition 1 remains true if we replace convexity by real-analyticity. Indeed, if  $c_M(q)$  denotes the rank of the Levi form of  $M$  at  $q \in M$  and  $\tilde{c}_M(p) = \limsup_{q \rightarrow p} c_M(q)$ , then, by

(C), through any  $q$  near  $p$  with  $c_M(q) = \tilde{c}_M(p)$  there exists a complex line  $(n - c_q)$ -dimensional complex plane that locally lies on  $M$ . Then one may find a  $(n - c_q)$ -dimensional complex plane with infinite order of contact with  $M$  at  $p$ . Since  $M$  is real-analytic, we conclude that this

plane lies on  $M$  near  $p$ . The real case follows analogously by using (R) instead of (C).

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